Extension: The Caliph's problem

Problem

A Caliph was exceptionally good at chess, so he challenged all visitors to play him. If he won, he said he would give the winner gold coins – one coin for the first square on the chessboard, two for the next, four for the next, eight for the next and so on, up to 2^{64} for the last square.

A young student turned up at the Caliph's palace wanting to take up the challenge. In addition, the student said that if she lost, she would give the Caliph one coin on the first square of an infinite chessboard, $\frac{1}{2}$ a coin on the next square, $\frac{1}{4}$ of a coin on the next square and so on forever.

How many coins was the Caliph proposing to give to the student if she beat him? How many coins was the young woman risking?

Problem steps

Step 1

Concentrate on the Caliph first. If you add the coins on successive squares you get 1, 3, 7, 15, 31, 63 and so on. These numbers should remind your class of the Towers of Hanoi problem. There, each of the number is of the form $2^S - 1$, where S is the number of squares considered to that point. This means the Caliph is offering $2^{64} - 1$ gold coins as a prize.

As usual, it is worthwhile creeping up on this one if students can't see the answer immediately.

Step 2

Can the students determine how many coins this is?

To work this out, students will need a calculator that can record a large number of decimal places, such as a CAS calculator or the calculator function on most computers.

The answer is 18 446 744 073 709 551 615; in scientific notation this is about 1.84×10^{19}.

(This is also the number of moves required to shift all 64 discs on the Tower of Hanoi, as shown in Level 3 Step 4.)

Step 3

How much was the young student risking?

Encourage your class to have a guess and write their guesses where they can be seen. We'll return to these guesses later.

Step 4

At this point it may be worthwhile to take a step backwards.

Can students prove that the number of coins is $2^{64} – 1$ *without* using the Towers of Hanoi result?

One way to do this is to write

$$S = 1 + 2 + 4 + 8 + \ldots + 2^{63}$$

Now double this:

$$2S = 2 + 4 + 8 + \ldots + 2^{63} + 2^{64}$$

Therefore:

$$\begin{aligned} S &= 2S – S \\ &= (2 + 4 + 8 + \ldots + 2^{63} + 2^{64}) – (1 + 2 + 4 + 8 + \ldots + 2^{63}) \\ &= 2^{64} – 1 \end{aligned}$$

Step 5

How many coins would there be if the Caliph had *trebled* the number of coins on successive squares, starting with one coin in the first square?

First get a closed form solution, using the same method as in Step 3. This time let S be as follows:

$$S = 1 + 3 + 9 + 27 + \ldots + 3^{63}$$

Now triple this:

$$3S = 3 + 9 + 27 + \ldots + 3^{63} + 3^{64}$$

Therefore:

$$\begin{aligned} S &= 3S – S \\ &= (3 + 9 + 27 + \ldots + 3^{63} + 3^{64}) – (1 + 3 + 9 + 27 + \ldots + 3^{63}) \\ &= 3^{64} – 1 \end{aligned}$$

This turns out to be 3 433 683 820 292 512 484 657 849 089 281, or 3.43×10^{30}.

Step 6

Let's return to the guesses from Step 3: How many coins was the young student risking?

Students should be able to calculate the answer using the method shown in Steps 4 and 5. However, they should start by calculating the answer for a finite chessboard with 64 squares.

Using the method above should give the sum as $2(1 - (\frac{1}{2})^{64})$ for 64 squares.

That turns out to be 1.99999999999999999989158.

Step 7

Does the student stand to lose much more as the number of squares on the chessboard increase? Work out what happens as the number of squares increases. Try 100 squares; try 200 squares.

The numbers get larger and large each time, but the total edges slowly closer and closer to 2 without ever getting bigger than 2.

To see this, we know from the method above that after S squares the total is $2(1 - (\frac{1}{2})^{S})$.

Now, as S gets bigger and bigger, $(\frac{1}{2})^{S}$ gets smaller and smaller. So as S approaches infinity, $(\frac{1}{2})^{S}$ approaches zero.

This means that as S approaches infinity, $2(1 - (\frac{1}{2})^{64})$ approaches 2 and never gets bigger than 2.

Where to from here?

- It's possible now to move off and explore geometric series. Change the Caliph's problem in any way you like, provided you keep multiplying successively by the same number throughout.