Extension: Incircles of triangles

Problem

Is it better to hide treasure by using the incircle of a triangle?

Problem steps

As noted in Level 4, an incircle is a circle inscribed in a figure so as to touch (but not cross) each side – that is, each side of the figure is a tangent to the circle.

Step 1

Start by discussing with your class how they could construct the incircle of a triangle. In Figure 1, how could you draw the circle if you just started off with the triangle *ABC*?

Or, when you are hiding treasure, would it be better to draw the circle first and then find some way to draw the triangle afterwards?

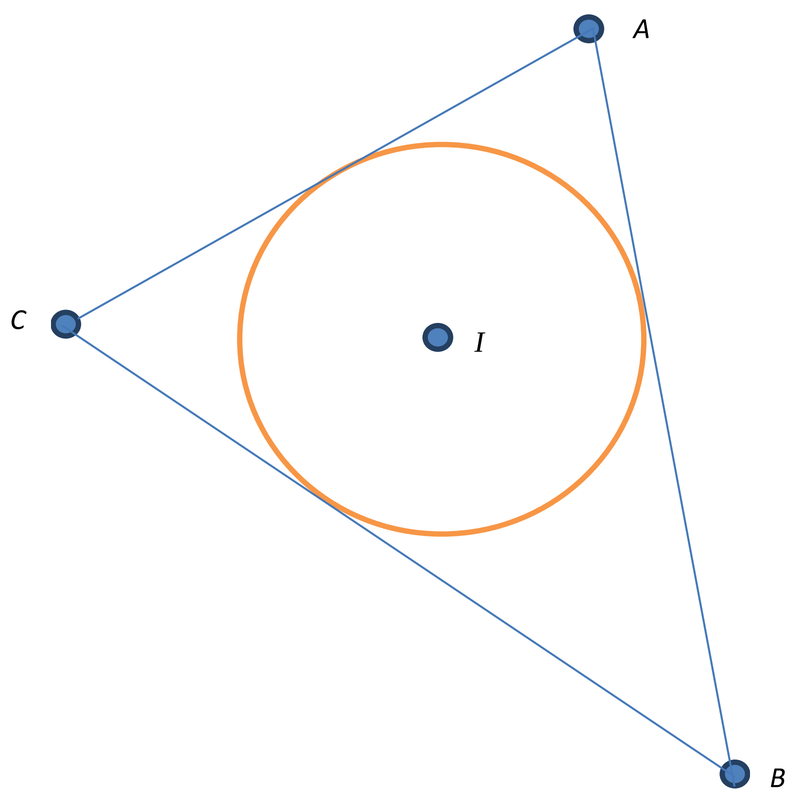

Figure 1: The incircle of a triangle and its incentre

Once your class appears to understand the task, let them work in pairs to try to produce either the circle from the triangle or vice versa. The key things they should be looking for:

1. what is the shortest distance from the incentre to any of the tangents
2. how do you find *I*, the centre of the incircle
3. how do you construct a tangent to a circle at a given point.

Step 2

Circle from the triangle: Look at Figure 2. In that diagram, the radius *IX* is perpendicular to *AB* and the radius *IY* is perpendicular to *BC*. This means that triangles *BXI* and *BYI* are congruent (RHS), and therefore angles *XBI* and *YBI* are equal.

If we could bisect the angle at *B*, then we could draw a line from *B* that passes though *I*. In the same way, the bisectors of the angles at *A* and *C* also go through *I*. Drawing these bisectors would give us the place to put the treasure as they would be equidistant from the pegs at *A*, *B* and *C*.

This process is what a treasure hunter should use to find the treasure using the three pegs at the vertices of a triangle.

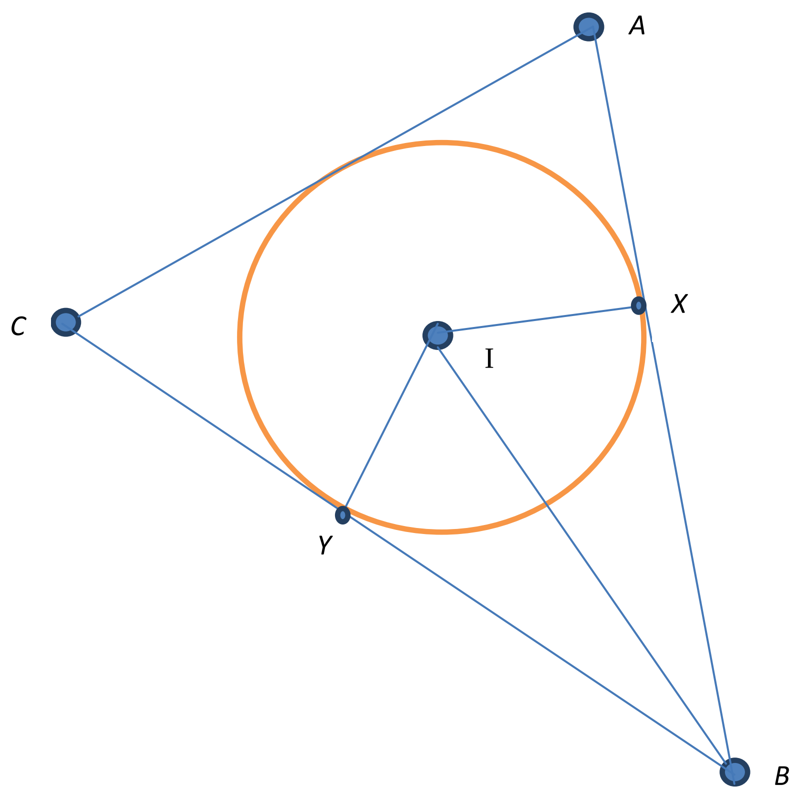

Figure 2: The incircle is on the bisectors of the angles of the triangle

Triangle from the circle: First we need to draw a tangent to the circle centre, *O*, at a given point, *P*. The diagram can be seen in Figure 3.

Extend *OP* to *H*. Now use the compass to find two points, *L* and *M*, the same distance away from *P* on *OH*. The perpendicular bisector of *LM* is the required tangent through P.

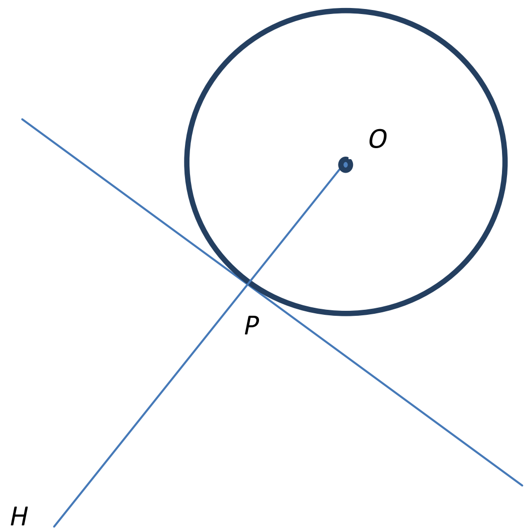

Figure 3: Constructing a tangent to the circle centre *O* at the point *P*

To get the triangle, choose three points, *X*, *Y* and *Z*, on the circle as the points where the three tangents meet the circle. The intersection points of every pair of tangents will give you the three vertices of the triangle. If the tangents don't produce a triangle, you need to move one of the points on the circumference until you do get a triangle.

By using tangents, how can you choose three points, *X*, *Y* and *Z*, on the circumference of a circle so that you *do* get a triangle with the original circle as its incircle?

Step 3

Your students can experiment here to find triples of points that do and don't have tangents that will form a triangle. Then get them to conjecture what the properties of *X*, *Y* and *Z* have to be.

Draw the diameter of the circle that passes through *X*. If *Y* is diametrically opposite to *X*, the tangents through *X* and *Y* will be parallel and thus will not meet. So choose any point *Y* in one of the semicircles formed by the diameter through *X*.

Suppose that *Z* is on the same semicircle as *Y*. If we look at the tangents through *Y* and *Z* these will both meet at a point. They will also meet the tangent though *X*, each at a different point (Figure 4).

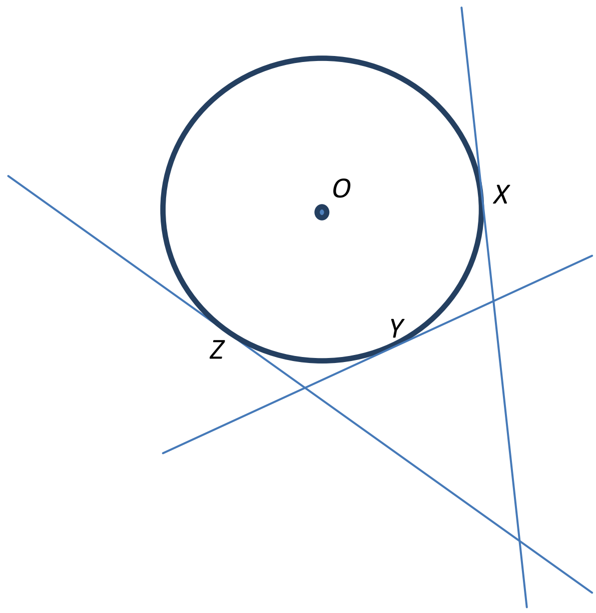

Figure 4: A triangle that doesn't have a given circle as its incircle

This circle is actually known as an *excircle* of the triangle.

Step 4

We have seen that a triangle has an incircle and a circumcircle. Is the incentre always the same point as the circumcentre? If this were so, then you could choose the easier of finding the incircle or the circumcentre of the triangle when you wanted to find the treasure.

This is another chance to experiment. Students should draw a variety of triangles and see if the *inradius* (radius of the incentre) and *circumradius* (the radius of the circumcircle) are always the same.

Start out with equilateral, isosceles and right-angled triangles. What do they find?

It would also be worth your students doing some searching on the web. Information on [Euler's theorem in geometry]() is particularly useful. This shows that if R is the circumradius and r is the inradius then $R^2 - 2rR = d^2$, where d is the distance between the two centres.

Step 5

Construct the incircle and the circumcircle of a given triangle. How many triangles have these circles as their incircle and circumcircle? Experiment.

Surprisingly, there are an infinite number. You can make another such triangle using any point on the circumcircle as a vertex.

Where to from here?

- Can your students find other ways to hide the treasure using pegs or some other fixed point system?