Extension: Animals on show

Problem

How many ways can 10 animals (cats and dogs) be put into two separate display cases?

Problem steps

If you've read all three books in the *Creative Activities in Mathematics* series, you'll recognise this is the same problem as 'Penny's Pet Shop' from Book 1. This activity starts from that same point, but takes a task for Foundation students and turns it into a complex Year 10 problem.

(By the way, cats are kept separate from dogs.)

Step 1

From first principles, the answer can quickly be worked out to be 11.

What if there were n animals? Would we get $n + 1$ ways of putting them in the display cases?

And what has this got to do with Pascal's triangle or binomial coefficients?

Step 2

Let's try a different problem.

How many solutions are there of the equation $x + y = 10$, where x and y are non-negative integers?

Unsurprisingly, the answer to this is also 11. Again, this seems to have nothing to do with binomial coefficients.

What if we looked at $x + y = n$? Would we get $n + 1$ possible solutions? Is $n + 1$ a binomial coefficient?

Step 3

What if we had four animals (cats, dogs and turtles) in three display cases? Would we get the same answer as the number of solutions of $x + y + z = 4$ where x, y, z are non-negative integers?

Let the class work on this in groups. They can do it in essentially two different ways. First, and most likely, they can directly count each situation and then see if the answers are the same. They should get 15 in each case.

Second, students can make a direct one-to-one comparison. Call x the number of cats, y the number of dogs and z the number of turtles. Then for every arrangement of the animals there is a solution to the equation, and for every solution of the equation there is a way of putting the animals in the display cases.

Step 4

What we have done so far raises two questions. Do we get the same count of animal arrangements and solutions provided only that the number of different types of animals is the same as the number of variables *and* the number of animals is the same as the number on the right side of the equation?

If that works out okay, is the answer a binomial coefficient? If it is, what is this number and how is it related to animals and display cases?

At this point it is time for the students to experiment in their groups and come up with a conclusion.

The counts are the same because of the one-to-one comparison argument. We show this in Figure 1. First, we have a variable for each display case. The number of animals in the first case corresponds to a value of *x*, and so on for all of the variables. Finally, the number of animals has to add up to the same number that the variables do.

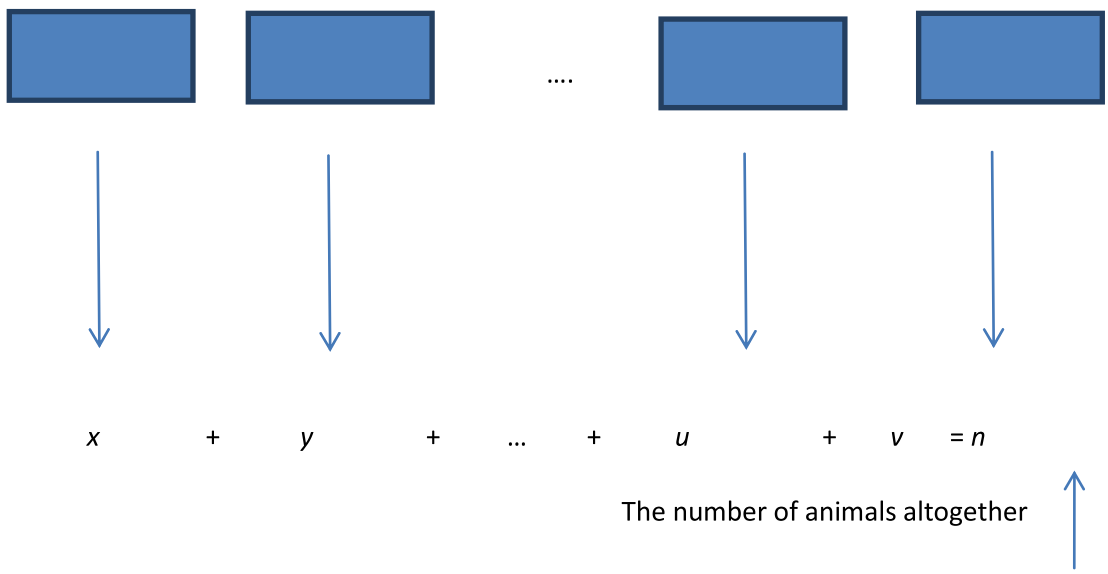

Figure 1: a one-to-one comparison

Step 5

But what does the answer turn out to be?

Suppose that Penny wanted to display 7 types of animals in 7 different display cases. In all she wanted to have 13 animals. How many ways could this happen?

The problem here is equivalent to finding the solutions of $t + u + v + w + x + y + z = 13$. How could students approach this?

One way is like this:

AA O AAA O A O O AA O A O AAAA

Here every A represents some animal and every O shows where a display case ends.

In our example this means we have two animals in the first case; three in the next; 1 in the next; none in the next; two in the next; one in the next; and four in the last.

To solve the problem overall we're looking at finding how many ways there are of choosing six Os from 19 Os and As. This is ${}^{19}C_6 = 27\,132$, which is far too many to do by writing them all out.

(By the way, for counting the number of solutions of the equation, the Os represent the plus signs and the As the different values of the variables.)

Step 6

Suppose that Penny is putting 6 types of animals in 6 cases and there are 11 animals overall. What is the probability that there are *at least* two cats?

This is sneaky. It looks like we are trying to find the number of solutions of $u + v + w + x + y + z = 11$ with a restriction.

Suppose u is the variable involved with cats. We need $u \geq 2$.

Now $u - 2 + v + w + x + y + z = 9$. Call this $u' + v + w + x + y + z = 9$.

We know the number of solutions for this - it is ${}^{14}C_5 = 2002$.

Where to from here?

- We have to solve equations like the ones above where there are c variables and their sum is equal to n, we will find that the binomial coefficient we need is why does the answer equal ${}^{n+c-1}C_{c-1}$ or equivalently ${}^{n+c-1}C_n$? Can your students proof this?
- What other situations can your students make up that can be solved this way? What other situations can your students make up that can be solved using binomial coefficients?
- What is the probability of having at least two cats in Steps 1, 2 and 3? Encourage your class to make up probability questions of their own.